PMM U.S.S.R.,Vol.52,No.1,pp.35-41,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

SEPARATING THE RAPID AND SLOW MOTIONS IN THE PROBLEMS OF THE DYNAMICS OF SYSTEMS OF RIGID BODIES AND GYROSCOPES*

S.V. BOGATYREV and V.A. SOBOLEV

A method for all asymptotic separation of rapid and slow motions based on the concepts of the Bogolyubov-Mitropol'skii theory of integral manifolds, is proposed for use in analysing systems of differential equations with small parameters for the derivatives which arise in the course of solving dynamic problems. The motion of a gyroscope with contactless suspension in a magnetic field is studied. The method of separating the motions enables the problem to be reduced to that of investigating a regular, finite-dimensional system of ordinary differential equations.

1. Formulation of the problem. We consider system whose equations of motion can be written in the form

$$x^{\cdot} = f(t, x, y, \varepsilon), \ \varepsilon y^{\cdot} = g(t, x, y, \varepsilon) \tag{1.1}$$

where x and y are vector variables, ε is a small positive parameter, and f and g are smooth vector functions.

Putting $\varepsilon = 0$ in (1.1), we obtain a so-called generating system

$$x^{\cdot} = f(t, x, y, 0), \ 0 = g(t, x, y, 0)$$

Let us assume that the second equation of this system has an isolated solution $y = h_0(t, x)$. The sufficient conditions for the existence of an integral manifold (IM) $y = h(t, x, \varepsilon), h(t, x, 0) = h_0(t,x)$ the motion along which takes place according to the equation

$$x^{\cdot} = f(t, x, h(t, x, \varepsilon), \varepsilon) \tag{1.2}$$

were given in /1-3/. Analysing this equation we can easily solve the problems of stability and of periodic solutions, as well as other problems concerned with the analytical study of the initial system.

The function h can be obtained in the form of an asymptotic expansion

$$h(t, x, \varepsilon) = h_0(t, x) + \varepsilon h_1(t, x) + \varepsilon^2 h_2(t, x) + \ldots$$

from the equation

$$\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial x} f(t, x, h, \varepsilon) = g(t, x, h, \varepsilon) \tag{1.3}$$

The solution of system (1.1) originating near the IM can be written as a sum of some solution lying in the IM, and a small, rapidly decaying supplement. Moreover, the problems of stability are equivalent for Eqs.(1.1) and (1.2). If, in particular, $f(t, 0, 0, \varepsilon) = 0$, $g(t, 0, 0, \varepsilon) = 0$, then $h(t, 0, \varepsilon) = 0$ and the zero solution of Eq.(1.1) is stable (asymptotically stable, unstable, or stable with respect to some of the variables) if and only if the zero solution of (1.2) has the same property. This means that the principle of reduction which makes it possible to reduce the investigation of the initial system of equations to that of Eqs.(1.2), holds for the IM $y = h(t, x, \varepsilon)$.

Such a principle was used in investigating the stability of the orientation of artificial satellites with dual rotation /2/, of gyroscopic systems /3/ and of systems of bodies with non-stationary internal masses /4/.

2. The scheme of separating the motions. The method of separating the rapid and slow motions based on the ideas of the theory of IM, consists of introducing new variables u and v by means of the formulas

$$\begin{aligned} x &= \beta + uH(t, u, v, \varepsilon) \\ y &= v + h(t, x, \varepsilon) = v + h(t, u + \varepsilon H(t, u, v, \varepsilon), \varepsilon) \end{aligned} \tag{2.1}$$

so as to obtain the equations

$$u^{\cdot} = F(t, u, \varepsilon), \ \varepsilon v^{\cdot} = G(t, u, v, \varepsilon) \tag{2.2}$$

*Prikl.Matem.Mekhan.,52,1,47-54,1988

the first of which is independent. The function h describes the IM $y = h(t, x, \varepsilon)$, and the function H describes an IM of some auxiliary extended system /5, 6/. The first equation of (2.2) describes the slow motions of the system in question, and the second the rapid motions. The function F is given by the equation

$$F(t, u, \varepsilon) = f(t, u, h(t, u, \varepsilon), \varepsilon) \tag{2.3}$$

If the matrix $A(t, x) = \partial g(t, x, h_0(t, x), 0)/\partial y$ has an inverse, then the function h can be calculated in the form of an expansion in powers of the small parameter from Eq.(1.3), with any degree of accuracy, using algebraic operations.

Let

$$Y(t, x, z, \varepsilon) = g(t, x, z + h, \varepsilon) - \varepsilon \frac{\partial h}{\partial t} - \varepsilon \frac{\partial h}{\partial x} f(t, x, z + h, \varepsilon)$$
$$(h = h(t, x, \varepsilon))$$

Then the function $H = H(t, u, v, \varepsilon)$ can be found in the form of an asymptotic expansion from the equation

$$\varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial u} F(t, u, \varepsilon) + \frac{\partial H}{\partial v} Y(t, u + \varepsilon H, v, \varepsilon) = \tag{2.4}$$
$$f(t, u + \varepsilon H, v + h(t, u + \varepsilon H, \varepsilon), \varepsilon) - f(t, u, h(t, u, \varepsilon), \varepsilon)$$

and the function G in (2.2) is given by the equation

$$G(t, u, v, \varepsilon) = Y(t, u + \varepsilon H(t, u, v, \varepsilon), v, \varepsilon) \tag{2.5}$$

It should be noted that the following inequalities hold for the functions H and G:

$$\| H(t, u, v, \varepsilon) \| \leqslant C \| v \|, \; \| G(t, u, v, \varepsilon) \| \leqslant C \| v \|$$

If the roots of characteristic equation $\det(A - \lambda E) = 0$ satisfy the inequality

$$\operatorname{Re} \lambda_i(t, x) \leqslant -2\alpha < 0 \tag{2.6}$$

then we have the following inequality for the variable v:

$$\| v(t, x) \| \leqslant K \exp(-\varepsilon^{-1} \alpha (t - t_0)), \; K > 0, \; t \geqslant t_0 \tag{2.7}$$

From relations (2.1), estimate (2.7) and the inequality for H and G, it follows that the solutions of system (1.1) can be written in the form

$$x = u + \varepsilon \varphi_1, \quad y = h(t, u, \varepsilon) + \varphi_2$$
$$\varphi_1 = H(t, u, v, \varepsilon), \quad \varphi_2 \equiv v + h(t, u + \varepsilon H(t, u, v, \varepsilon), \varepsilon) - h(t, u, \varepsilon)$$
$$\| \varphi_i \| \leqslant C \| y(t_0) - h(t_0, x(t_0), \varepsilon) \| \exp(-\varepsilon^{-1} \alpha (t - t_0))$$
$$C > 0, \; t \geqslant t_0 \; (i = 1, 2)$$

Relations (2.1) show that the solutions of system (1.1) represent a non-linear superposition of the slow variable u and the rapid variable v. The relations enable us to split not only the equations, but also the initial conditions. If the initial conditions $x(t_0) = x_0$, $y(t_0) = y_0$ are given for Eqs.(1.1), then from the second equation of (2.1) it follows that $v(t_0) = v_0 = y_0 - h(t_0, x_0, \varepsilon)$, and $u(t_0) = u_0$ where u_0 is found from the equation

$$x_0 = u_0 + \varepsilon H(t_0, u_0, v_0, \varepsilon)$$

We note that condition (2.6) does not hold for the equations of motion of mechanical systems with low dissipation, and in particular for the gyroscopic systems /3, 6, 7/. Nevertheless the method of IM can be used to separate the rapid and slow motions in the case of such systems also.

3. Rapid and slow motions of gyroscopic systems. The equations of motion of a wide class of gyroscopic system can be written in the following form /7/:

$$dx/dt = y, \; \varepsilon d(Ay)/dt = -[G + \varepsilon B] y + \varepsilon R + \varepsilon Q, \tag{3.1}$$
$$R = {}^1/_2 [\partial (Ay)/\partial x]^T y$$

Here x is an n-dimensional vector of generalized coordinates, A is a symmetric, positive definite matrix, G is a skew symmetric matrix of gyroscopic forces, B is a symmetric matrix of the dissipative forces, Q is the vector of generalized forces, ε is a small positive parameter, and A, B, G, G^{-1}, Q are functions of t and x, and we assume that they are bounded and have sufficient numbers of partial derivatives in t and x.

System (3.1) has an IM $y = \varepsilon h(t, x, \varepsilon)$, and the motion along it is described by the equation

$$dx/dt = \varepsilon h(t, x, \varepsilon) \tag{3.2}$$

The initial variables x and y are connected with the new slow variable u and rapid variable v by the relations

$$x = u + \varepsilon H(t, u, v, \varepsilon) \tag{3.3}$$
$$y = v + \varepsilon h(t, x, \varepsilon) = v + \varepsilon h(t, u + \varepsilon H(t, u, v, \varepsilon), \varepsilon)$$

Equations analogous to (1.3) and (2.4) yield the following approximate expressions:

$$h(t, x, \varepsilon) = h_1(t, x) + \varepsilon h_2(t, x) + \varepsilon^2 \ldots$$
$$H(t, u, v, \varepsilon) = H_1(t, u, v) + \varepsilon H_2(t, u, v) + \varepsilon^2 \ldots$$
$$h_1 = G^{-1}Q, \quad h_2 = -G^{-1}[Bh_1 + \partial(Ah_1)/\partial t]$$
$$H_1 = -G^{-1}Av, \quad H_2 = -[(\partial G^{-1}/\partial t)A - G^{-1}B]G^{-1}Av + O(\| v \|^2)$$

The matrices A, B, G and the function Q depend, in the expressions for h_1, h_2, on t and x and in the expressions for H_1, H_2, on t and u.

The expressions for the variables u and v have the form

$$du/dt = \varepsilon h(t, u, \varepsilon), \quad \varepsilon d(Av)/dt = -(G + \varepsilon B)v + \varepsilon R(t, u, v) + \varepsilon^2 R_1(t, u, v, \varepsilon) \tag{3.4}$$
$$A = A(t, u + \varepsilon H), \quad B = B(t, u + \varepsilon H), \quad G = G(t, u + \varepsilon H)$$
$$P(t, x, y, \varepsilon) = \frac{1}{2}\left(\frac{\partial A}{\partial x} y\right)^T h + \frac{1}{2}\left(\frac{\partial A}{\partial x} h\right)^T y - \left(\frac{\partial Ah}{\partial x}\right) y$$
$$h = h(t, u + \varepsilon H, \varepsilon), \quad H = H(t, u, v, \varepsilon)$$
$$R_1(t, u, v, \varepsilon) = P(t, u + \varepsilon H(t, u, v, \varepsilon), v, \varepsilon)$$

The first equation of (3.4) describes the slow precession oscillations of the gyroscopic system, and the second describes the rapid nutation oscillations. The first formula of (3.3) shows that the vector of generalized coordinates x represents the superposition of the precession and nutation oscillations.

Thus, using the substitution (3.3), we have succeeded in separating the system of Eqs.(3.1) into two Eqs.(3.4), obtaining at the same time approximate expressions for the functions h and H.

If in particular the initial conditions $x(t_0) = x_0, y(t_0) = \varepsilon y_0$ are given for Eqs.(3.1), then $v(t_0) = \varepsilon v_0 = \varepsilon(y_0 - h(t_0, x_0, \varepsilon))$ and initial condition $u(t_0) = u_0$ for the first equation of (3.4) is found from the equation

$$x_0 = u_0 + \varepsilon H(t_0, u_0, \varepsilon v_0, \varepsilon)$$

in the form of the asymptotic expansion

$$u_0 = x_0 + \varepsilon^2 G^{-1}(t_0, x_0) A(t_0, x_0)[y_0 - G^{-1}(t_0, x_0) Q(t_0, x_0)] + \varepsilon^3 \ldots$$

It is important to note that the problem discussed above are closely connected with the problem of the admissibility of using the equations of the precession theory /3, 6-10/. The following equations are precessional for the first equation of (3.4):

$$[G(t, x) + \varepsilon B(t, x)]\, dx/dt = \varepsilon Q(t, x) \tag{3.5}$$

or, in equivalent form,

$$dx/dt = \varepsilon [G(t, x) + \varepsilon B(t, x)]^{-1} Q(t, x)$$

It can be confirmed that the right-hand side of this equation is identical with the right-hand side of Eq.(3.2), apart from terms of the order of $O(\varepsilon)$ inclusive for the non-autonomous system, and up to terms of order $O(\varepsilon^2)$ inclusive for the autonomous system.

Taking into account the fact that under the assumptions made above for system (3.1) the nutational oscillations decay and the reduction principle holds, we can conclude that the "truncated" Eqs.(3.2) or (3.5) can be used instead of the initial Eqs.(3.1).

4. Some generalizations. The IM method can be used successfully to investigate the problems of control theory /5, 11/, systems with several small parameters /12/, systems with random parameters /13/, and a wide range of other problems in mechanics. The application of this method to systems with distributed parameters is of particular interest. We need only show that a certain boundary value problem for the partial differential equations can be formulated as an operator equation in a suitable Hilbert space. Such a procedure can often be successfully carried out for many problems encountered in practice. In particular, we shall do this below for certain problems of dynamics of a conducting rigid body in a magnetic field.

We consider the systems of equations of the form

$$x^{\cdot} = f(t, x, y, \varepsilon),\ \varepsilon y^{\cdot} + Ay = g(t, x, y, \varepsilon) \tag{4.1}$$

where A is an unbounded operator in some Hilbert space Y. Under certain conditions which ensure the stability of the operator A, system (4.1) has an IM $y = h(t, x, \varepsilon)$ which is exponentially stable and satisfies the reduction principle. We note that the conditions of stability of the operator A consist of the fact that the spectrum of this operator lies in some sector situated in the right half-plane of C, and the resolvent has, outside this sector as $|\lambda| \to \infty$, an asymptotic estimate of the form $O(|\lambda|^{-1})$.

It can be shown that the method of separating the motions admits of a generalization to systems of the form (4.1), i.e. there exists a change of variables

$$x = u + H(t, u, v, \varepsilon),\ y = v + h(t, x, \varepsilon) \tag{4.2}$$

which reduces system (4.1) to a system of the form

$$u^{\cdot} = F(t, u, \varepsilon),\ \varepsilon v^{\cdot} + Av = G(t, u, v, \varepsilon) \tag{4.3}$$

in which the rapid and slow motions are already separated. The functions h, H in (4.2) and F, G in (4.3) are found using a method analogous to that given in Sect.2. Practical difficulties arise in determining the asymptotic expansion of the function H, and hence in constructing the function G, but it can be done in certain important special cases.

Let us consider the following special case of system (4.1):

$$x^{\cdot} = f_0(t, x) + f_1(t, x)y,\ \varepsilon y^{\cdot} + Ay = \varepsilon g(t, x) \tag{4.4}$$

and specify for this system the following boundary conditions:

$$x(t_0) = x_0,\ y(t_0) = y_0 \tag{4.5}$$

Using the algorithms given in Sect.2 for constructing the asymptotic expansions of the functions $h(t, x, \varepsilon)$ and $H(t, u, v, \varepsilon)$, we can obtain the following asymptotic representation:

$$\begin{aligned} &H(t, u, v, \varepsilon) = -\varepsilon f_1(t, u)(A^{-1}v) + \varepsilon^2 \ldots \\ &h(t, x, \varepsilon) = \varepsilon A^{-1} g(t, x) - \varepsilon^2 A^{-2}\left[\frac{\partial g(t, x)}{\partial t} + \frac{\partial g(t, x)}{\partial x} f_0(t, x)\right] + \varepsilon^3 \ldots \end{aligned} \tag{4.6}$$

Then the change of variable (4.2) reduces problem (4.4), (4.5) to the form

$$\begin{aligned} &u^{\cdot} = f_0(t, u) + \varepsilon[f_1(t, u) A^{-1} g(t, u)] - \\ &\quad \varepsilon^2\left[f_1(t, u) A^{-2}\left(\frac{\partial g(t, u)}{\partial t} + \frac{\partial g(t, u)}{\partial u} f_0(t, u)\right)\right] + \varepsilon^3 \ldots \\ &\varepsilon v^{\cdot} + Av = -\varepsilon^2 A^{-1}\left[\frac{\partial g(t, u)}{\partial u} f_1(t, u) v\right] + \varepsilon^3 \ldots \end{aligned} \tag{4.7}$$

$$u(t_0) = u_0,\ v(t_0) = v_0 \tag{4.8}$$

where we have the following asymptotic representations for u_0, v_0:

$$\begin{aligned} &u_0 = x_0 + \varepsilon f_1(t_0, x_0) A^{-1} y_0 + \varepsilon^2 \ldots \\ &v_0 = y_0 - \varepsilon A^{-1} g(t_0, x_0) + \varepsilon^2 \ldots \end{aligned} \tag{4.9}$$

5. On the motion of a conducting rigid body about the centre of mass in a magnetic field. We shall consider the problem of the rotation of a conducting rigid body about the centre of mass, in a uniform magnetic field. We assume that the body is a uniform, isotropic ideal magnetic material of conductivity λ and magnetic permeability μ, and the characteristic time of diffusion of the vorticity of the field within the body is substantially less than the characteristic time of variation in an external magnetic field in a coordinate system attached to the body.

The problem is solved by simultaneous investigation of the equations of motion of the rigid body about the centre of mass, and the equations of electrodynamics written in the quasistationary approximation

$$I\omega^{\cdot} + \omega \times I\omega = N,\ \Gamma^{\cdot} = -\Omega\Gamma \tag{5.1}$$

$$\begin{aligned} &\frac{4\pi}{c^2} B^{\cdot} + \frac{1}{\lambda\mu} \operatorname{rot}\operatorname{rot} B = 0, \quad \operatorname{div} B = 0 \quad (r \in V) \\ &\operatorname{rot} B^{(0)} = 0,\ \operatorname{div} B^{(0)} = 0 \quad (r \in V_0) \\ &B_\tau = B_\tau^{(0)}|_S, \quad \mu B_n = B_n^{(0)}|_S; \quad B^{(0)}|_\infty = \Gamma B^\infty \end{aligned} \tag{5.2}$$

Here I is the inertia tensor of the body, ω is the angular velocity vector within the body, Γ is the matrix of transition from the inertial absolute coordinate system to a system rigidly attached to the body, B^∞ is the external magnetic field vector in the absolute coordinate system, Ω is a matrix placed in correspondence with the vector ω so that the relation $\Omega r = \omega \times r$, will hold for all $r \in R^3$, B and $B^{(0)}$ are the magnetic field vectors in the body V and outside the body V_0, $B_\tau|_S$ and $B_n|_S$ are the tangential and normal components of the vector B at the surace S of the body. The moment N of the forces acting on the body in a magnetic field is expressed in terms of the Maxwell stress tensor T according to the formula

$$N = \int_S [r \times Tn]\, ds, \quad T = \left\{T_{ij} = \frac{1}{4\pi}\left(B_i^{(0)} B_j^{(0)} - \frac{1}{2}\| B^{(0)} \|^2 \delta_{ij}\right)\right\}$$
$$(i, j = 1, 2, 3)$$

We take as the unit of time the characteristic time of variation in an external magnetic field in the associated coordinate system, and the characteristic dimension of the body as the unit of space variables. Then, under the assumptions made above, the parameter $\varepsilon = 4\pi/c^2$ in Eqs. (5.2) will be small.

Earlier /14-17/ a similar formulation was used to study the problem of the motion of a conductor in a magnetic field.

To transform problem (5.1), (5.2) to the form (4.4), we carry out the following change of variables:

$$B = b + (E + \Psi)\Gamma B^\infty, \quad B^{(0)} = b^{(0)} + (E + \Psi^{(0)})\,\Gamma B^\infty$$

which leads to a null condition at infinity and preserves the homogeneity of the conditions on S. Here E is the unit matrix and the matrices Ψ and $\Psi^{(0)}$ admit of the representation

$$\Psi = \| \nabla\psi_1, \nabla\psi_2, \nabla\psi_3 \|, \quad \Psi^{(0)} = \| \nabla\psi_1^{(0)}, \nabla\psi_2^{(0)}, \nabla\psi_3^{(0)} \|$$

where the functions $\psi_j, \psi^{(0)}_j$ $(j = 1, 2, 3)$ are solutions of the problem

$$\Delta\psi_j = 0 \ (r \in V), \ \Delta\psi_j^{(0)} = 0 \ (r \in V_0) \ (j = 1, 2, 3)$$
$$\psi_j = \psi_j^{(0)}|_S, \quad \mu\frac{d\psi_j}{dn} - \frac{d\psi_j^{(0)}}{dn}\bigg|_S = (1 - \mu)(e_j)_n|_S; \quad \psi_j|_\infty = 0$$

where e_j $(j = 1, 2, 3)$ are unit vectors of the associated coordinate system. Using the new variables we can rewrite to system (5.2) in the form

$$\varepsilon b^\cdot + \frac{1}{\lambda\mu}\operatorname{rot}\operatorname{rot} b = -\varepsilon(E + \Psi)(\Gamma B^{\cdot\infty} + \Gamma B^\infty \times \omega) \tag{5.3}$$
$$\operatorname{div} b = 0 \ (r \in V); \ \operatorname{rot} b^{(0)} = 0, \ \operatorname{div} b^{(0)} = 0 \ (r \in V_0)$$
$$b_\tau = b_\tau^{(0)}|_S, \quad \mu b_n = b_n^{(0)}|_S, \quad b^{(0)}|_\infty = 0$$

Then we shall have the following expression for the moment of the forces:

$$N = [J\Gamma B^\infty \times \Gamma B^\infty] + \left(\frac{\mu - 1}{4\pi}\int b\, dr + \frac{1}{8\pi}\int [r \times \operatorname{rot} b]\, dr\right) \times \Gamma B^\infty$$
$$J = \frac{\mu - 1}{4\pi}\int \Psi\, dr$$

Here and henceforth the integration will be carried out over the volume V of the body.

Now problem (5.1), (5.3) can be rewritten in the form of system (4.4), provided that the operator A and the functions f_0, f_1, g are defined in a suitable manner.

Indeed, let us introduce the notation $x' = \omega$, $x'' = \Gamma$, $X' = \Omega$, $y = b$ and denote by $x = (x', x'')$ the vector composed, successively, of the elements of the vector x' and the elements of the column vectors of the matrix x''. Let

$$f_0(t, x) = (f_0'(t, x), f_0''(t, x)) = (I^{-1}(Ix' \times x' + Jx''B^\infty \times x''B^\infty), -X'x'')$$
$$f_1(t, x)\, y = (f_1'(t, x)\, y, f_1''(t, x)\, y) =$$
$$\left(I^{-1}\left[\left(\frac{\mu - 1}{4\pi}\int y\, dr + \frac{1}{8\pi}\int [r \times \operatorname{rot} y]\, dr\right) \times x''B^\infty\right], 0\right)$$
$$g(t, x) = -(E + \Psi)(x''B^{\cdot\infty} + x''B^\infty \times x')$$

We define the operator $A: Y \to Y$ by the equation $Ay = (\lambda\mu)^{-1}\operatorname{rot}\operatorname{rot} y$, and assume that the space Y consists of the vectors y, square summable over V and satisfying the condition of solenoidality

$$\operatorname{div} y = 0 \tag{5.4}$$

The domain of definition of the operator A consists of the vectors y belonging to the Sobolev space $H^2(V)$, satisfying the condition of solenoidality (5.4) and possessing a continuation

$\Pi y \in H^2(V_0)$ with the properties

$$\operatorname{div} \Pi y = 0, \ \operatorname{rot} \Pi y = 0 \ (r \in V_0)$$
$$y_\tau = (\Pi y)_\tau|_S, \ \mu y_n = (\Pi y)_n|_S$$

The generalized derivatives are used here in defining the operators div, rot. It was shown in /18/ that the operator A defined in this manner will satisfy the required conditions of stability.

It is now clear that after such a reformulation, problem (5.1),(5.3) can be rewritten in the form of system (4.6).

Eq.(4.7) for the slow motions $u = (\omega, \Gamma)$ will have the form

$$\begin{aligned} &I\omega^{\cdot} + \omega \times I\omega = J\Gamma B^\infty \times \Gamma B^\infty + \varepsilon P_1 [\Gamma B^{\cdot\infty} + \Gamma B^\infty \times \omega] \times \\ &\quad \Gamma B^\infty + \varepsilon^2 P_2 [\Gamma B^{\cdot\cdot\infty} + 2\Gamma B^{\cdot\infty} \times \omega + (\Gamma B^\infty \times \omega) \times \omega + \\ &\quad \Gamma B^\infty \times I^{-1} (I\omega \times \omega + J\Gamma B^\infty \times \Gamma B^\infty)] \times \Gamma B^\infty + \ldots \\ &\Gamma^{\cdot} = -\Omega\Gamma \end{aligned} \tag{5.5}$$

Here J is the tensor characterizing the magnetizability of the body in an external magnetic field, and $P_1, P_2, \ldots$ are the magnetic polarizability tensors of the body /15/. All these tensors are determined only by the form of the body and its electrical and magnetic characteristics. Their determination reduces to solving certain classical stationary boundary value problems.

The properties of the IM of slow motions imply that for every solution $(\omega(t), \Gamma(t), b(t))$ of system (5.1), (5.3) a solution $(\omega_*(t), \Gamma_*(t))$ of system (5.5) can be found such, that the following inequality will hold for all $t \geqslant t_0$:

$$\| \omega(t) - \omega_*(t) \| + \| \Gamma(t) - \Gamma_*(t) \| \leqslant K \exp(-\varepsilon^{-1}\alpha(t - t_0)) \tag{5.6}$$

If the solution $(\omega(t), \Gamma(t), b(t))$ is determined by the initial conditions, **then** the initial conditions for the solution $(\omega_*(t), \Gamma_*(t))$ of system (5.5) satisfying inequality (5.6) can be found from the relations (4.9) in a unique manner.

The expression on the right-hand side of the first equation of (5.5) represents an asymptotic expansion of the slow part of the moment of the forces N in powers of the small parameter ε. In the special case of $\lambda = 1, \mu = 1$ the expansion will be identical with the expansion of the moment of the forces obtained in /15/.

In fact, the system of Eqs.(5.5) describes the motion of the body about the centre of mass under the action of the moment of forces generated by the eddy currents and magnetization of the body in the external magnetic field outside some initial time interval.

It is important to note that by separating the rapid and slow motions, we have succeeded in reducing the problem described by a system of partial differential equations with singular perturbations, to the study of a regular, finite-dimensional system of ordinary differential equations.

The main results of this paper were presented at the Sixth All-Union Session on Theoretical and Applied Mechanics /19/.

The authors thank V.V. Strygin for useful advice and criticism.

REFERENCES

1. MITROPOL'SKII YU.A. and LYKOVA O.B., Integral Manifolds in Non-linear Mechanics. Moscow, Nauka, 1973.
2. STRYGIN V.V. and SOBOLEV V.A., Effect of the geometrical and kinematic parameters and the dissipation of energy on the stability of the orientation of the artificial satellites with dual rotation. Kosmich. Issledovaniya, 14, 3, 1976.
3. SOBOLEV V.A. and STRYGIN V.V., On the admissibility of the transition to the precessional equations of gyroscopic system. Isv. Akad. Nauk SSSR, MTT, 5, 1978.
4. SOBOLEV V.A. and STRYGIN V.V., **Asymptotic** methods in problems of stabilizing rotating bodies using passive dampers. Isv. Akad. Nauk SSSR, MTT, 5, 1977.
5. SOBOLEV V.A., Integral manifolds and decomposition of singularly perturbed systems, Systems Control Letters. 5, 3, 1984.
6. SOBOLEV V.A., Rapid and slow motions of gyroscopic systems. Periodica Polytechnica. **Electrical** Engineering, Budapest, 29, 1, 1985.
7. MERKIN D.R., Gyroscopic Systems. Moscow, Gostekhizdat, 1956.
8. ISHLINSKII A.YU., Mechanical of Gyroscopic Systems. Moscow, Izd-vo Akad. Nauk SSSR, 1963.
9. BUTENIN I.V., KLIMOV D.M., LUNTS YA.L. and STEPANENKO I.P., Non-linear problems of the theory of gyroscopic systems. In: Development of Mechanics of Gyroscopic and Inertial Systems. Moscow, Nauka, 1973.

10. KOBRIN A.I., MARTYNENKO YU.G. and NOVOZHILOV I.V., On the precession equations of gyroscopic systems. PMM 40, 2, 1976.
11. KOKATOVIC P.V., Control theory in the 80's, Trend in feedback design: Proc. 9th World Congr. of IFAC. Budapest, 1984.
12. PENDYUKHOVA N.V., SOBOLEV V.A. and STRYGIN V.V., The motion of a rigid body with a gyroscope and mobile mass. Izv. Akad. Nauk SSSR, MTT, 3, 1986.
13. SOBOLEV V.A., Singularly perturbed stochastic equations. Fourth International Vil'nyus Conference on the Theory of Probability and Mathematical Statistics. Vil'nyus, In-t. Matematiki i Kibernetiki of the Academy of Sciences of the LitSSR, 3, 1985.
14. GOLUBKOV V.V., The moment of forces in a magnetic field. Kosmich. Issledovaniya, 10, 1, 1972.
15. KOBRIN A.I. and MARTYNENKO YU.G., Motion of a conducting rigid body about the centre of mass in a slowly rotating magnetic field. Dokl. Akad, Nauk SSSR, 261, 5, 1981.
16. KOBRIN A.I., Asymptotic solution of the problem of the motion of a rigid body in an magnetic field. Differents. Uravneniya, 21, 10, 1985.
17. BOGATYREV S.V. and STRYGIN V.V., On the stability of the stationary motions of a conducting rigid body about the centre of mass in a magnetic field. Izv. Akad, Nauk SSSR, MTT, 5, 1986.
18. LADYZHENSKAYA O.A. and SOLONNIKOV V.A., On the principle of linearization and on invariant manifolds for problems of magnetohydrodynamics. Zap. Nauch. Seminarov LOMI, 38, 1973.
19. BOGATYREV S.V., SOBOLEV V.A. and STRYGIN V.V., On the separation of rapid and slow motions in problems of the dynamics of systems of rigid bodies and gyroscopes. Sixth All-Union Session on Theoretical and Applied Mechanics. Tashkent, Nats. Kom-t SSSR, po Teoret. in Prikl. Mekhanike, 1986.

Translated by L.K.

PMM U.S.S.R.,Vol.52,No.1,pp.41-48,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

STABILITY DIAGRAMS OF THE PERIODIC MOTIONS OF A PENDULUM WITH AN OSCILLATING AXIS*

Z.S. BATALOVA and G.V. BELYAKOVA

Periodic rotations of a pendulum with a harmonically oscillating axis of suspension are studied analytically and numerically. General regularities in their bifurcation diagrams are established, depending on the evenness of the numbers characterizing the number of rotations of the pendulum and the period of oscillations of the axis of suspension.

The phenomenon of the dynamic stability of the upper position of the pendulum and the effect of vibrational excitation and of the maintenance of its rotations have found wide application in modern devices and mechanisms /1-3/. The mathematical models of the motions of a parametrically excited pendulum in the form of non-linear, non-autonomous differential equations, taking resistance forces into account, were studied by analytical methods and a number of periodic modes were investigated numerically (see /2/ where a survey of the investigations and a bibliography are given, and also /4-6/). In the Hamiltonian case the periodic motions of a rotational body have not been studied before.

The present paper deals with periodic rotations of a parametrically excited non-linear oscillator, without taking the dissipation into account. The Cesari method is used to obtain the generating solutions, a number of periodic rotations of a single type are established and their stability is studied in the case when the values of two parameters are small. A number of solutions of practical interest are continued numerically into the domain of large values of the parameters. The bifurcation diagrams

**Prikl.Matem.Mekhan.*,52,1,55-63,1988